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Two different kinds of time delays in a stochastic system

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Abstract. The steady state properties of a noise-driven bistable system are investigated when there are two different kinds of time delays existed in the deterministic and fluctuating forces respectively. Using the approximation of the probability density approach, the delayed Fokker-Planck equation is obtained. The stationary probability distribution (SPD) and the variance of the system are derived. It is found that the time delay τ in the deterministic force can reduce the fluctuations while the time delay β in the fluctuating force can enhance the fluctuations. Numerical simulations are presented and are in good agreement with the approximate theoretical results.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 02.30.Ks Delay and functional equations – 02.50.Ey Stochastic processes

1 Introduction

In recent years, the effects of time delays on stochastic dynamical systems have attracted much attention in various fields, such as laser systems with optical delayed feedback [1–8], stochastic resonance with delayed interactions [9–12], chemical surface reactions [13], population dynamics [14], the spread of the infectious diseases [15], etc. In these complex systems, time delay plays an important role in the dynamical properties of these systems. The time delay usually arises from the finite transmission time of the matter, energy and information. In many cases, the time delay can be regarded as a useful description of the systems involving a reaction chain or a transport process.

For stochastic dynamical systems without time delay, the steady state and dynamical properties have been widely investigated [16–28]. More recently, the effects of the time delay existing in the deterministic force of a stochastic system have been discussed [29–36]. However, the time delay appeared in both deterministic and fluctuating forces needs to be investigated. The main problem caused by the time delay in the stochastic system is that the appropriate analytical result is difficult to be derived due to the non-Markovian process appeared in the system.

In this paper, the steady state properties of a stochastic bistable system are investigated when two different kinds of time delays exist in the deterministic and fluctuating forces respectively. In Section 2, the approximation of the probability density approach is applied to obtain the delayed Fokker-Planck equation. In Section 3, the analytical expressions of the stationary probability distribution and the variance of the system are derived. The effects of the two different kinds of time delays are discussed. In Section 4, the numerical simulations are presented and compared with the analytical results. A discussion concludes the paper.

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2 Theoretical analysis

For a delayed stochastic differential equation, some approximation methods need to be applied in order to obtain analytical results. One of the approximations is the probability density approach. If this approximation is employed, the non-Markov process can be reduced to a Markov process.

2.1 Delayed Fokker-Planck equation

A one-dimensional stochastic delayed differential equation driven by two coupled white noise terms $\Gamma(t)$ and $\eta(t)$ follows the Langevin equation

$$\frac{dx(t)}{dt} = h(x(t), x(t-\tau)) + g(x(t), x(t-\beta))\Gamma(t) + Q\eta(t). \tag{1}$$

Where the parameter τ denotes the delay time in the deterministic force while β denotes the delay time in the random force, Q is a constant, $\Gamma(t)$ and $\eta(t)$ are Gaussian white noises with zero mean and correlations

$$\begin{aligned} \langle \Gamma(t)\rangle &= \langle \eta(t)\rangle = 0, \\ \langle \Gamma(t)\Gamma(t')\rangle &= 2P'\delta(t-t'), \\ \langle \eta(t)\eta(t')\rangle &= 2P\delta(t-t'), \\ \langle \Gamma(t)\eta(t')\rangle = \langle \Gamma(t')\eta(t)\rangle &= 2\lambda\sqrt{P'P}\delta(t-t'). \end{aligned} \tag{2}$$

Here P' and P are the intensities of the noise terms $\Gamma(t)$ and $\eta(t)$ respectively, λ denotes the coupling strength between two noise terms.

If the approximation of the probability density approach is employed [29,32–34], equation (1) can be rewritten as

$$\frac{dx(t)}{dt} = h_{\mathrm{eff}}(x(t)) + g_{\mathrm{eff}}(x(t))\Gamma(t) + Q\eta(t), \tag{3}$$

here,

$$\begin{aligned} h_{\mathrm{eff}}(x) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x, x_\tau) \\ &\quad \times P(x_\tau, t-\tau; x_\beta, t-\beta|x,t)\, dx_\tau dx_\beta, \\ g_{\mathrm{eff}}(x) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x, x_\beta) \\ &\quad \times P(x_\tau, t-\tau; x_\beta, t-\beta|x,t)\, dx_\tau dx_\beta. \end{aligned} \tag{4}$$

In equation (4), $P(x_\tau, t-\tau; x_\beta, t-\beta|x,t)$ denotes the conditional distribution of $x(t)$. Thus, the stochastic delayed differential equation can be approximately reduced to the ordinary stochastic equation. The non-Markovian process induced by the time delays can be transformed to Markovian process. Meanwhile, equation (3) can be equivalently transformed into a stochastic differential equation

$$\frac{dx(t)}{dt} = h_{\mathrm{eff}}(x) + G_{\mathrm{eff}}(x)\epsilon(t), \tag{5}$$

with

$$\begin{aligned} &\langle \epsilon(t)\epsilon(t')\rangle = 2\delta(t-t'), \\ &G_{\mathrm{eff}}(x) = \sqrt{P' g_{\mathrm{eff}}^2 + 2\lambda\sqrt{P'P} g_{\mathrm{eff}} Q + PQ^2}. \end{aligned} \tag{6}$$

From equations (5) and (6), the delayed Fokker-Planck equation corresponding to equations (1) and (2) can be derived as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x)P(x,t)] + \frac{\partial^2}{\partial x^2}[B(x)P(x,t)], \tag{7}$$

with

$$\begin{aligned} A(x) &= h_{\mathrm{eff}}(x) + G_{\mathrm{eff}}(x)\frac{dG_{\mathrm{eff}}(x)}{dx}, \\ B(x) &= G_{\mathrm{eff}}^2(x). \end{aligned} \tag{8}$$

Here $P(x,t) = \langle \delta(x - x(t))\rangle$ denotes the probability density of the stochastic process. Consequently, the stationary probability distribution (SPD) can be obtained

$$P_{\mathrm{st}}(x) = \frac{N}{G_{\mathrm{eff}}(x)} \exp\int_{-\infty}^{x} dx' \frac{h_{\mathrm{eff}}(x')}{G_{\mathrm{eff}}^2(x')}, \tag{9}$$

where N is the normalization constant, $G_{\mathrm{eff}}(x)$ and $h_{\mathrm{eff}}(x)$ can be evaluated from equations (4) and (6).

2.2 Two different kinds of time delays in a bistable system

The approximate theory can be applied to a bistable system. For a time-delayed bistable system with coupling between additive and multiplicative noise terms, the dimensionless Langevin equation follows

$$\frac{dx(t)}{dt} = x(t-\tau) - x^3(t) + x(t-\beta)\Gamma(t) + \eta(t), \tag{10}$$

here $\Gamma(t)$ is the multiplicative noise, $\eta(t)$ is the additive noise, both $\Gamma(t)$ and $\eta(t)$ are the same as that in equation (2). Comparing equations (10) and (1), the following relations are obtained

$$h(x(t), x(t-\tau)) = x_\tau - x^3;\; g(x(t), x(t-\beta)) = x_\beta;\; Q = 1. \tag{11}$$

Since x_τ and x_β are independent variables, $P(x_\tau, t-\tau; x_\beta, t-\beta|x,t)$ can be given by

$$\begin{aligned} &P(x_\tau, t-\tau; x_\beta, t-\beta|x,t) = P(x_\tau, t-\tau|x,t)P(x_\beta, t-\beta|x,t) \\ &= \sqrt{\frac{1}{2\pi G^2(x,x)\tau}}\sqrt{\frac{1}{2\pi G^2(x,x)\beta}} \\ &\times \exp\left(-\frac{[x_\tau - (x + (x - x^3)\tau)]^2}{2G^2(x,x)\tau} - \frac{[x_\beta - (x + x\beta)]^2}{2G^2(x,x)\beta}\right). \end{aligned} \tag{12}$$

The normalized conditional distribution function $P(x_\tau, t-\tau|x,t)$ and $P(x_\beta, t-\beta|x,t)$ can be expressed as follows [37]

$$\begin{aligned} P(x_\tau, t-\tau|x,t) &= \sqrt{\frac{1}{2\pi G^2(x,x)\tau}} \\ &\quad \times \exp\left(-\frac{[x_\tau - (x + h(x,x)\tau)]^2}{2G^2(x,x)\tau}\right), \\ P(x_\beta, t-\beta|x,t) &= \sqrt{\frac{1}{2\pi G^2(x,x)\beta}} \\ &\quad \times \exp\left(-\frac{[x_\beta - (x + g(x,x)\beta)]^2}{2G^2(x,x)\beta}\right), \end{aligned} \tag{13}$$

where $h(x,x) = x - x^3$, $g(x,x) = x$, $G^2(x,x) = P'x^2 + 2\lambda\sqrt{P'P}x + P$. After the integration, one has

$$\begin{aligned} h_{\mathrm{eff}}(x) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x, x_\tau) \\ &\quad \times P(x_\tau, t-\tau; x_\beta, t-\beta|x,t)\, dx_\tau dx_\beta \\ &= (1+\tau)(x - x^3), \\ g_{\mathrm{eff}}(x) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x, x_\beta) \\ &\quad \times P(x_\tau, t-\tau; x_\beta, t-\beta|x,t)\, dx_\tau dx_\beta \\ &= (1+\beta)x, \end{aligned} \tag{14}$$

and

$$G_{\mathrm{eff}}(x) = \sqrt{P'(1+\beta)^2x^2 + 2\lambda\sqrt{P'P}(1+\beta)x + P}. \tag{15}$$

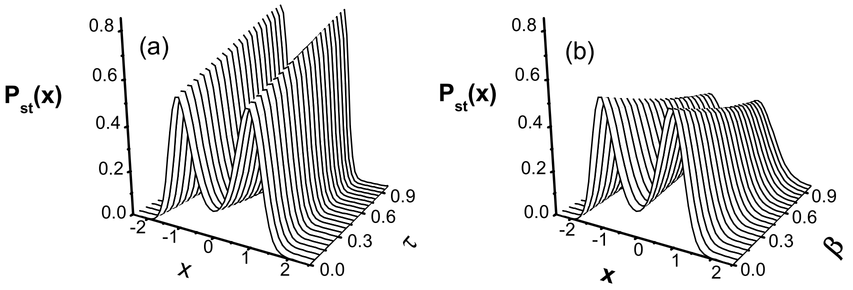

Fig. 1. The stationary probability distribution function $P_{\rm st}(x)$ is plotted as a function of x, τ, and β respectively. The parameters are dimensionless and are chosen as $P' = P = 0.1, \lambda = 0$. (a) $P_{\rm st}(x)$ as a function of x and τ with $\beta = 0$. (b) $P_{\rm st}(x)$ as a function of x and β with $\tau = 0$.

3 Stationary properties in a bistable system

The effects of the two different time delays can be evaluated from the stationary probability distribution function $P_{\rm st}(x)$ and the variance σ_x^2 of the variable x of the bistable system. The stationary probability distribution function and the variance can be calculated from equation (9).

3.1 Stationary probability distribution

The stationary probability distribution (SPD) can be expressed by

$$P_{\rm st}(x) = \frac{N}{G_{\rm eff}(x)} \exp\left[\frac{-\phi(x)}{P'}\right], \tag{16}$$

here, $\phi(x)$ is the rectified potential function. From equations (9), (14) and (15), $\phi(x)$ reads

$$\begin{aligned}\phi(x) &= \kappa_1 x + \kappa_2 x^2 + \kappa_3 \arctan\frac{[P'(1+\beta)x + \lambda\sqrt{P'P}]}{\sqrt{P'P(1-\lambda^2)}} \\ &\quad + (\kappa_4 \ln[P'(1+\beta)^2 x^2 + 2\lambda\sqrt{P'P}(1+\beta)x + P)]),\end{aligned} \tag{17}$$

with

$$\begin{aligned}\kappa_1 &= \frac{-2\lambda\sqrt{P'P}}{P'(1+\beta)^3}(1+\tau), \qquad \kappa_2 = \frac{1+\tau}{2(1+\beta)^2}, \\ \kappa_3 &= \frac{\lambda(1+\tau)[P'(1+\beta)^2 + P(3-4\lambda^2)]}{P'(1+\beta)^3\sqrt{(1+\beta)^2(1-\lambda^2)}}, \\ \kappa_4 &= \frac{(1+\tau)[4P\lambda^2 - P - P'(1+\beta)^2]}{2P'(1+\beta)^4}.\end{aligned} \tag{18}$$

If there is no time delay with $\tau = \beta = 0$, equation (16) is reduced to the previous expression obtained in a bistable system [21]. The three dimensional curves of the stationary probability distribution (SPD) $P_{\rm st}(x)$ as a function of the variable x are plotted in Figure 1 when the delay time τ and β are varied.

The SPD as a function of the variable x and delay time τ is plotted in Figure 1a. From Figure 1a, it is seen that the SPD of the system exhibits a symmetric bimodal structure when τ is increased. The height of the two peaks and the depth of the valley increase when τ is increased. The position of the two peaks shifts slightly away from $x = \pm 1$. The position of the valley is kept at $x = 0$.

The SPD as a function of the variable x and the delay time β is plotted in Figure 1b. From Figure 1b, it is seen that similar structure as that in Figure 1a is obtained. The height of the two peaks and the depth of the valley decrease when the delay time β is increased. The position of the two peaks shifts closer to $x = 0$ while the position of the valley is still kept constant at $x = 0$. It is clear that the effect of β on SPD is contrary to that of τ. That is, the effect of time delay in deterministic force is opposite to that in fluctuation force.

3.2 Variance of variable x

From the stationary probability distribution of equation (16), the nth moment of the variable x can be calculated as follows

$$\langle x^n \rangle = \int_{-\infty}^{+\infty} x^n P_{\rm st}(x) dx. \tag{19}$$

Then the variance can be given by

$$\begin{aligned}\sigma_x^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\ &= \int_{-\infty}^{+\infty} x^2 P_{\rm st}(x) dx - \left(\int_{-\infty}^{+\infty} x P_{\rm st}(x) dx\right)^2.\end{aligned} \tag{20}$$

The variance can be obtained by numerical integration of equation (20). The three dimensional curves of the variance σ_x^2 are plotted in Figure 2 as a function of the multiplicative noise intensity P' when the delay time τ and β are varied.

The variance σ_x^2 as a function of the delay time τ and multiplicative noise intensity P' is plotted in Figure 2a. From Figure 2a, it is seen that the variance increases monotonously when P' is increased. It decreases monotonically when τ is increased. It is clear that multiplicative noise P' can enhance the fluctuation while the delay time τ can suppress the fluctuation in a delayed bistable system.

The variance σ_x^2 as a function of the delay time β and the multiplicative noise intensity P' is plotted in Figure 2b. From Figure 2b, it is seen that the variance increases monotonically when both P' and β are increased. The delay time β in the fluctuation force can enhance the fluctuation in the system. For small value of P', σ_x^2 is increased slowly with β. While for large value of P', σ_x^2 is increased very fast with β. Comparison of Figures 2a and 2b

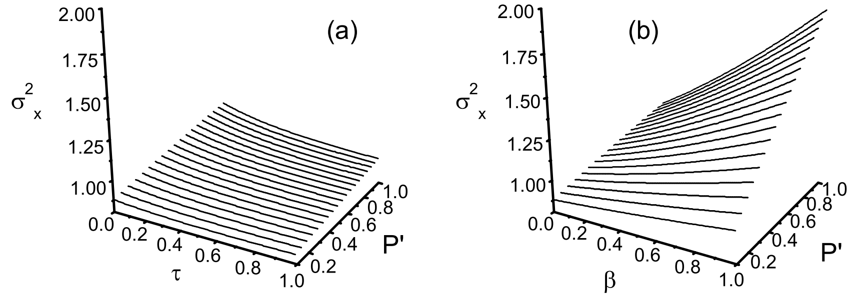

Fig. 2. The variance σ_x^2 is plotted as a function of τ, β, and P' respectively. The parameters are dimensionless and are chosen as $P = 0.3, \lambda = 0$. (a) σ_x^2 as a function of τ and P' with $\beta = 0$. (b) σ_x^2 as a function of β and P' with $\tau = 0$.

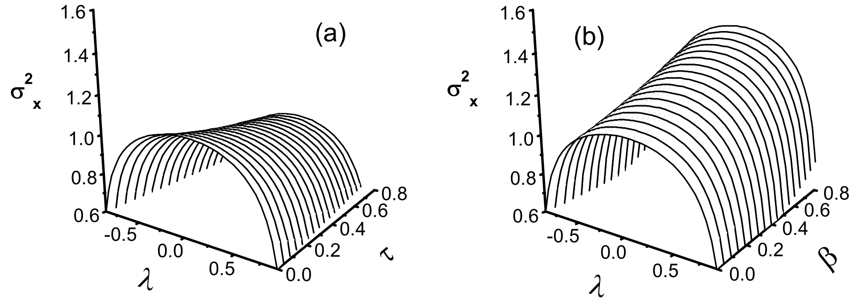

Fig. 3. The variance σ_x^2 is plotted as a function of τ, β, and λ respectively. The parameters are dimensionless and are chosen as $P' = P = 0.5$. (a) σ_x^2 as a function of λ and τ with $\beta = 0$. (b) σ_x^2 as a function of λ and β with $\tau = 0$.

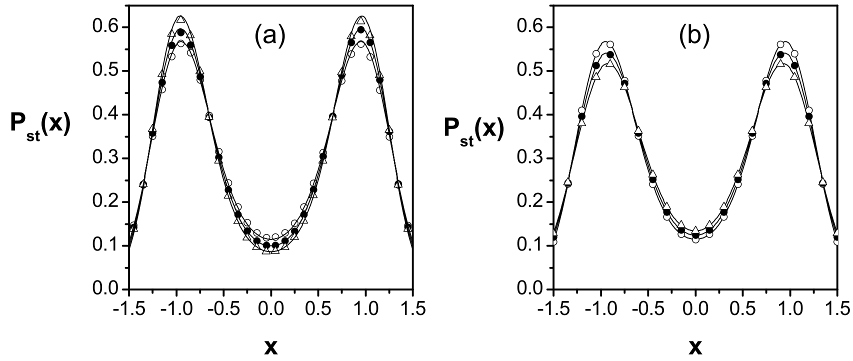

Fig. 4. The numerical simulation of the stationary probability distribution function $P_{\rm st}(x)$ is plotted as a function of x when τ and β are varied respectively. The symbols represent the numerical simulations and the solid lines represent the approximate analytical results. The parameters are dimensionless and are chosen as $P' = P = 0.1, \lambda = 0$. (a) $P_{\rm st}(x)$ as a function of x when $\beta = 0$ and $\tau = 0$ (○); 0.1 (●); 0.2 (△). (b) $P_{\rm st}(x)$ as a function of x when $\tau = 0$ and $\beta = 0$ (○); 0.1 (●); 0.2 (△).

shows that the effect of β on the variance is opposite to that of τ.

The three dimensional curves of the variance σ_x^2 are plotted in Figure 3 as a function of the coupling strength λ between the additive and multiplicative noise terms when the delay time τ and β are varied.

The variance as a function of the coupling strength λ and delay time τ is plotted in Figure 3a. From Figure 3a, it is seen that the maximum value of the variance σ_x^2 is located at $\lambda = 0$. The variance decreases symmetrically at two sides of $\lambda = 0$. When τ increases, the height of the peak is decreased while the position of the peak remains at $\lambda = 0$.

The variance as a function of the coupling strength λ and delay time β is plotted in Figure 3b. From Figure 3b, it is seen that the structure of σ_x^2 is similar to that shown in Figure 3a. The height of the peak in σ_x^2 is increased when β is increased.

From Figures 2 and 3, it is clear that the delay time τ in deterministic force can suppress the fluctuation while the delay time β in the fluctuation force can enhance the fluctuation in a bistable system. The delay time β plays a more important role than τ in the fluctuation of a bistable system.

4 Numerical simulation

To check the validity of the approximation method of the probability density approach in a bistable system, the numerical simulation is employed. The simulation can be performed by integrating the stochastic time-delayed Langevin equation (10). The Box-Mueller algorithm is used to generate Gaussian noise. Using Euler procedure, the time-discrete numerical data are calculated with the integration step of $\Delta t = 0.005$. An ensemble of $N = 10^5$ realizations of x is obtained from equation (10) by numerical calculations. For each realization of x the cycle is repeated for 1000 times. Accordingly, the stationary probability distribution $P_{\rm st}(x)$ and the variance σ_x^2 can be obtained.

The results of numerical simulations of the SPD as a function of x are plotted in Figure 4 when the time delays τ and β are varied respectively. In Figure 4a, $P_{\rm st}(x)$

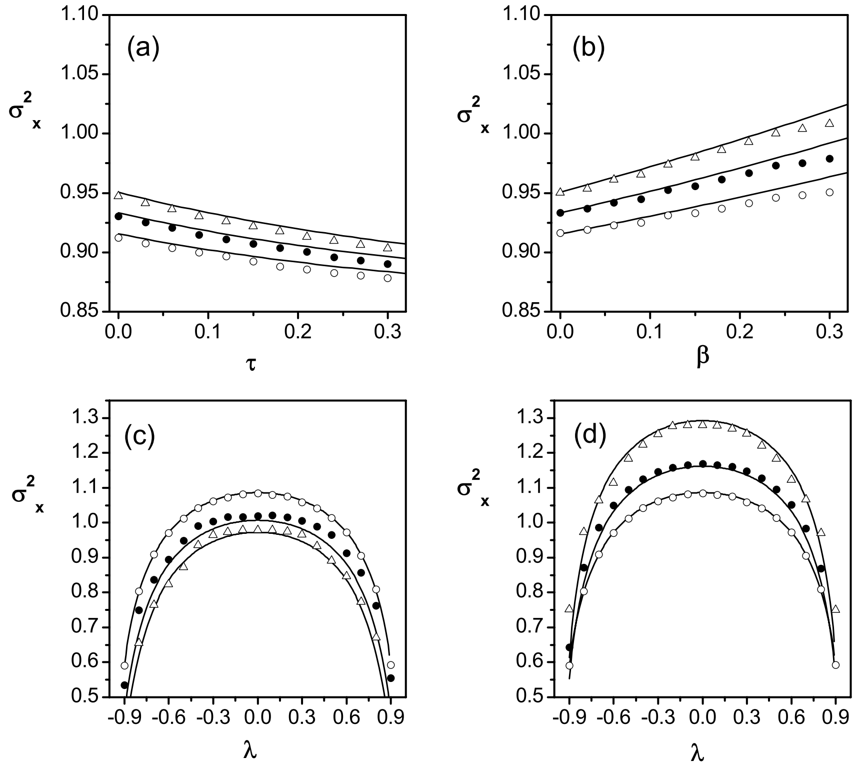

Fig. 5. The numerical simulation of the variance σ_x^2 is plotted as a function of τ, β and λ respectively. The symbols represent the numerical simulations and the solid lines represent the approximate analytical results. (a) σ_x^2 as a function of τ when $\lambda = \beta = 0$, $P = 0.3$ and $P' = 0.2$ (○); 0.25 (●); 0.3 (△). (b) σ_x^2 as a function of β when $\lambda = \tau = 0$, $P = 0.3$ and $P' = 0.2$ (○); 0.25 (●); 0.3 (△). (c) σ_x^2 as a function of λ when $P' = P = 0.5$, $\beta = 0$, and $\tau = 0$ (○); 0.3 (●); 0.5 (△). (d) σ_x^2 as a function of λ when $P' = P = 0.5$, $\tau = 0$, and $\beta = 0$ (○); 0.3 (●); 0.5 (△).

is plotted as a function of x when τ is varied. While in Figure 4b, $P_{\rm st}(x)$ is plotted as a function of x when β is varied. From Figure 4, it is clearly seen that the approximate theoretical results of the SPD are in good agreement with the numerical simulations.

The results of the numerical simulations of the variance σ_x^2 are plotted in Figure 5 as a function of τ, β and the coupling strength λ, respectively. In Figures 5a and 5b, σ_x^2 is plotted as a function of τ and β when the multiplicative noise intensity P' is varied. From Figures 5a and 5b, it is seen that the approximate theoretical results are slightly higher than the numerical simulations. For small values of τ and β, i.e., for small time delay, the approximate theoretical results are consistent with the numerical simulations. However, the deviation becomes large when τ and β are increased. In Figures 5c and 5d, σ_x^2 is plotted as a function λ when τ and β are varied. From Figures 5c and 5d, it is clear that excellent agreement between approximate theory and numerical computation is obtained.

5 Conclusion

The steady state properties of a noise-driven bistable system are investigated when there are two different kinds of time delays existed in the deterministic and fluctuating forces respectively. Using the approximation of the probability density approach, the non-Markovian process induced by the time delays is transformed to the Markovian process. The approximate delayed Fokker-Planck equation is obtained. The analytical expressions of the stationary probability distribution and the variance are derived.

The effects of the two time delays are different. It is found that the time delay τ in the deterministic force can increase the height of the peaks in SPD while the time delay β in the fluctuating force can decrease the height of the peaks in SPD. That is, the delay time τ can suppress while β can enhance the fluctuation. The different effects of the two time delays in the system is mainly due to the different forms of their existence. One appears in the deterministic force while another appears in the fluctuation force. Numerical simulations are in good agreement with the approximate theoretical results.

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